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Chap 2. Basic properties of Fourier series

2.1. Definitions and examples.

Def. Let f be a $\mathbb{C}$-valued function defined on $[-\pi, \pi]$.

We say that f is Riemann integrable (or simply, integrable) if both the real part, and the imaginary part of f are integrable on $[-\pi, \pi]$.

Def. Let $f : [-\pi, \pi] \to \mathbb{C}$ be integrable. The Fourier series of f is defined as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx},$$

where $$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

We call $\hat{f}(n)$ the n-th Fourier coefficient of f.

Example 1. Let $f : [-\pi, \pi] \to \mathbb{C}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi], \\ -1 & \text{if } x \in [-\pi, 0). \end{cases}$$

Write out the Fourier series of f on $[-\pi, \pi]$.

Solution:

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{\pi} 1\, dx + \frac{1}{2\pi}\int_{-\pi}^{0} (-1)\, dx = 0.$$

For $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\, dx - \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_0^{\pi} - \frac{1}{2\pi}\left[\left. \frac{e^{-inx}}{-in} \right|_{-\pi}^{0} \right]$$

$$= \frac{1}{2\pi} \frac{e^{-in\pi}-1}{-in} - \frac{1}{2\pi} \frac{1-e^{+in\pi}}{-in}$$

$$= \frac{1}{2\pi} \frac{e^{-in\pi}+e^{in\pi}-2}{(-in)}$$

$$= \begin{cases} \dfrac{-2i}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence the Fourier series of f is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{-2i}{(2n+1)\pi} e^{i(2n+1)x}$$

Example 2. $f(x) = x$ on $[-\pi, \pi]$.

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = \frac{1}{2\pi} \cdot \frac{x^2}{2} \Big|_{-\pi}^{\pi} = 0$$

For $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx$$

$$= \frac{1}{2\pi} \left(x \frac{e^{-inx}}{-in} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{e^{-inx}}{in} dx \right)$$

$$= \frac{1}{2\pi} \left(\frac{\pi e^{-in\pi} + \pi e^{in\pi}}{-in} + \frac{e^{-inx}}{n^2} \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{(-1)^n}{n}.$$

Hence $f(x) \sim \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n}{n} e^{inx}$.

2.2. Uniqueness of Fourier series.

Q: Suppose f and g are two integrable functions on $[-\pi, \pi]$

Such that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$.

Do we have $f \equiv g$?

The answer is "No". Because if f and g differ only at countable many points, then $\hat{f}(n) \equiv \hat{g}(n)$.

However, the following result says that if $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then f and g coincide at continuity points.

Thm 2.1 Suppose that f is integrable on the circle such that

$\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then

$f(x_0) = 0$ if f is continuous at x_0.

Pf. We first assume that f is a real valued function.

Suppose $x_0 \in (-\pi, \pi)$ is a continuity point of f.

We need to show $f(x_0) = 0$.

Suppose on the contrary that $f(x_0) \neq 0$.

WLOG, assume that $f(x_0) > 0$.

Choose a small $\delta > 0$ such that

$$f(x) > \frac{f(x_0)}{2} \quad \text{if} \quad |x - x_0| < \delta.$$

Our idea is to construct some trigonometric polynomial $P_k(x)$ such that $\int_{-\pi}^{\pi} f(x) \cdot P_k(x)\, dx \neq 0$, which leads to a contradiction.

Take a small $\varepsilon > 0$ such that

$$\cos(x - x_0) < 1 - \frac{3\varepsilon}{2} \quad \text{if} \quad \delta < |x - x_0| < \pi$$

Take a small $0 < \eta < \delta$ such that

$$\varepsilon + \cos(x - x_0) > 1 + \frac{\varepsilon}{2} \quad \text{if} \quad |x - x_0| < \eta.$$

Consider $P_k(x) = \left(\varepsilon + \cos(x - x_0)\right)^{2k}$.

$$\int_{-\pi}^{\pi} f(x)\cdot P_k(x)\,dx = \int_{-\pi+x_0}^{\pi+x_0} f(x)\cdot P_k(x)\,dx$$

$$= \int_{|x-x_0|<\eta} + \int_{\eta\leq|x-x_0|\leq\delta} + \int_{\delta<|x-x_0|<\pi} f(x)\,P_k(x)\,dx$$

$$= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}),$$

where

$$(\mathrm{I}) = \int_{|x-x_0|<\eta} f(x)\left(\varepsilon+\cos(x-x_0)\right)^{2k} dx$$

$$\geq \frac{f(x_0)}{2}\cdot\left(1+\frac{\varepsilon}{2}\right)^{2k}\cdot 2\eta \quad \to +\infty \quad \text{as } k\to+\infty.$$

$$(\mathrm{II}) = \int_{\eta\leq|x-x_0|\leq\delta} f(x)\left(\varepsilon+\cos(x-x_0)\right)^{2k} dx \geq 0.$$

$$|(\mathrm{III})| \leq \int_{\delta<|x-x_0|<\pi} |f(x)|\cdot\left(\varepsilon+\cos(x-x_0)\right)^{2k} dx$$

$$\leq \int_{\delta<|x-x_0|<\pi} \left(\sup|f|\right)\cdot\left(1-\frac{\varepsilon}{2}\right)^{2k} dx$$

$$= \left(\sup|f|\right)\cdot\left(1-\frac{\varepsilon}{2}\right)^{2k}\cdot(2\pi-2\delta) \quad \to 0 \quad \text{as } k\to+\infty.$$

It follows that as $k \to +\infty$,

$$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \longrightarrow +\infty.$$

Hence

$$\int_{-\pi}^{\pi} f(x)\, P_k(x)\, dx \neq 0 \quad \text{when } k \text{ is large enough.}$$

This leads to a contradiction, as $P_k(x)$ is a trigonometric polynomial.

(Recall a trigonometric polynomial is of the form

$$\sum_{n=-N}^{N} c_n e^{inx}$$
).

Next we assume that f is a $\mathbb{C}$-valued function.

Write $f(x) = u(x) + i\,v(x)$, where u, v are the real and imaginary parts of f. It is easy to check that

$$\hat{u}(n) = \hat{v}(n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Now both u and v are cts at x_0. So $u(x_0) = 0 = v(x_0)$.

Hence $f(x_0) = 0$. ▨

Corollary 2.2. If f is cts on the circle such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

2.3 Convergence of Fourier series

Assume $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

Let $S_N f(x) = \sum_{|n| \le N} c_n e^{inx}$

Q: $S_N f(x) \to f(x)$?

Thm 2.3. Assume that f is cts on the circle and

$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$.

Then $S_N f(x) \rightrightarrows f(x)$ on the circle.

Pf. Since $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$, by Weierstrass M-Test Thm,

$$S_N f(x) \rightrightarrows g(x) \text{ on the circle} \qquad (*)$$

for some cts function g on the circle.

By $(*)$, for each $n \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int S_N f(x)\, e^{-inx}\, dx \to \hat{g}(n) \quad \text{as } N \to +\infty.$$

However, $\frac{1}{2\pi}\int S_N f(x)\, e^{-inx}\, dx = \hat{f}(n)$ when $N \geq |n|$.

So $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$.

Since both f, g are cts on the circle, by Cor 2.2,

$f \equiv g$. By $(*)$, $S_N f(x) \rightrightarrows f(x)$ on the circle. $\blacksquare$